Explicitly solvable Coulomb and hydrogen atom systems with barrier on flat and curved spaces

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Introduction

Superintegrable systems in classical and quantum physics are those systems of maximum possible symmetry and this forces explicit analytic and algebraic solvability. For most physical systems the differential equations describing system behavior can only be solved numerically. However, superintegrable systems can be solved explicitly without need for numerical approximation. In this research, two particularly interesting superintegrable systems are studied: Kepler-Coulomb problems with barrier, E16 and the analog of this system on a sphere,S7. They have been little studied and have important physical meanings.

Method

This project will use the algebraic and geometric tools of superintegrability theory, based on symmetry, to solve these systems, rather than the traditional tools of mathematical physics. The structure equations of the system will be found by the Poisson algebra. These equations will lead us to the explicit trajectories of Kepler problem in both E16 and S7.

Result

E16:

- Constants of Motion: L_1,L_2, and H
- structure relation which is the Poisson bracket of L_1 and L_2:

$$R^2 = 4(L_1+a_3)\left(L_1H-L_2{}^2\right)+a_1{}^2L_1+2a_1a_2L_2+a_2{}^2H\ .$$

- Trajectories in polar coordinate:

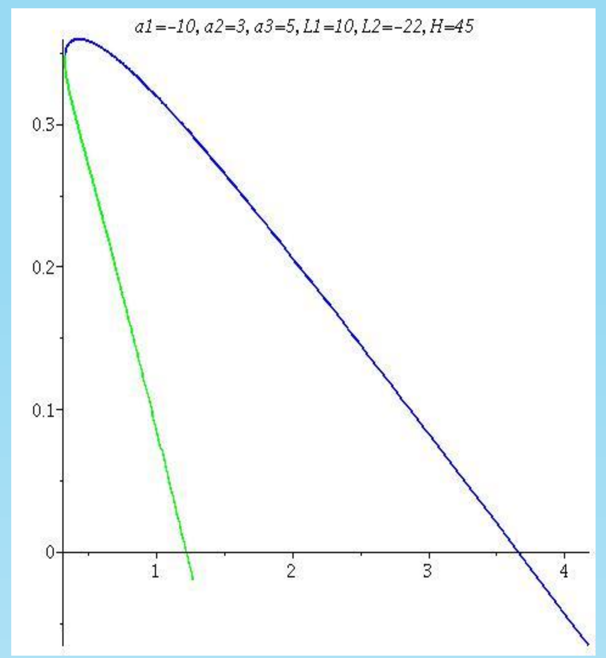
$$\mathrm{r}=\frac{N(sin\theta)\pm 2\sqrt{S(sin\theta)}}{D(sin\theta)} \quad \text{where}$$

$$\mathrm{N}=(a_1a_2-4a_3L_2-4L_1L_2)sin\theta-2a_1L_1-2a_2L_2$$

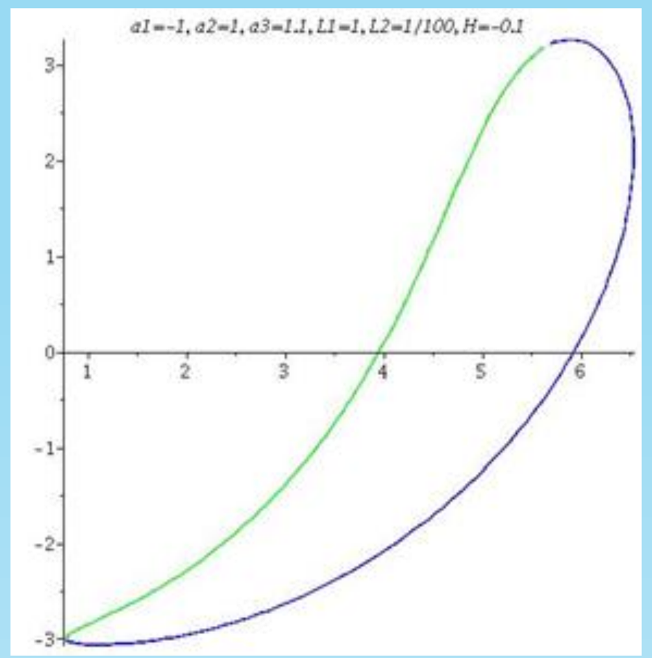
$$\mathrm{S}=[-(L_1+a_3)sin^2\theta+\ L_1-\ a_2sin\theta]\ R^2$$

$$\mathrm{D}=[4\mathrm{H}(L_1+a_3)+a_1{}^2]sin^2\theta+(4a_1L_2+4a_2H)sin\theta-4HL_1+4L_2{}^2$$

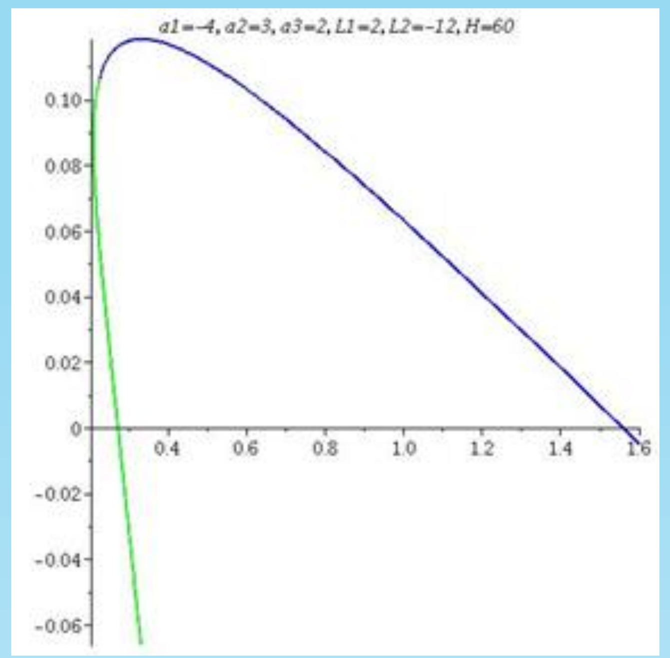
- Case I: $a_3>a_2$, unbounded trajectories correspond to H≥0, and bounded orbits happen at H<0
- Case II: $a_3<a_2$, here if L_1<0, the trajectories crash into the lower y axis and perpendicular to the axis at impact, and if L_1>0, The trajectories do not necessarily intersect the negative y-axis.
- Plots:



Case I positive energy



Case I negative energy



Case II L_1>0

S7:

- Constants of Motion: L_1,L_2, and H
- structure relation which is the Poisson bracket of L_1 and L_2:

$$R^2=-4L_1{}^2(L_1+H-4a_2)-4L_1\left(L_2{}^2+a_2H\right)+a_1{}^2H+(a_3{}^2-a_1{}^2)L_1-2a_1a_3L_2-a_2a_3{}^2$$

- Trajectories in polar coordinate $s_1=rcos\theta, s_2=rsin\theta, s_3=\pm\sqrt{1-r^2}$:

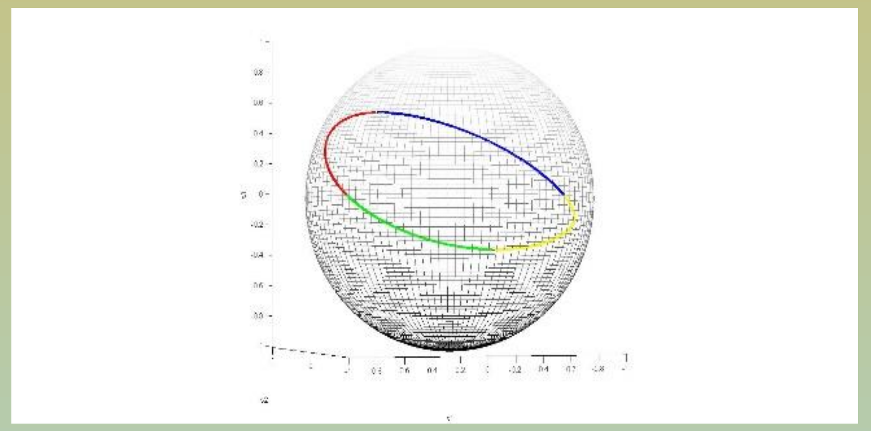
$$\cos\theta=\frac{N(r)\pm 2\sqrt{S(r)}}{D(r)} \quad \text{where}$$

$$\mathrm{N}=a_1a_3\sqrt{1-r^2}+2a_3rL_2-2Hra_1+4L_1L_2\sqrt{1-r^2}+2rL_1a_1$$

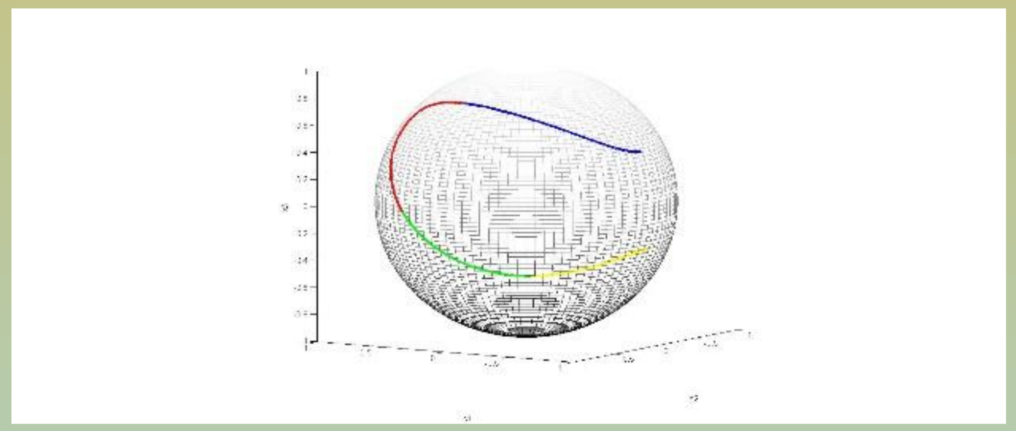
$$\mathrm{S}=(\mathrm{H}r^2-\sqrt{1-r^2}ra_3-L_1)R^2$$

$$\mathrm{D}=\mathrm{r}(4\mathrm{H}L_1-4L_1{}^2+a_3{}^2)$$

- Case I: $a_2>a_1>0$, all trajectories are bounded and periodic
- Case II: $a_1>a_2$, then the portion $-1 < s_1 < -a_2/a_1$ of the s_1 axis becomes attractive and the trajectory crashes.
- Plots:



Case I



Case II

Interesting Case of E16: metronome orbits

- Case of setting $R^2=0$
- equation of orbits: $\beta^2(x^2+y^2)=\alpha^2y^2\pm 2\alpha y+1,\ \alpha^2+\ \beta^2>0.$
- these metronome orbits are conic sections:

$$\beta^2>\alpha^2>0 \qquad ellipse$$
$$0<\beta^2<\alpha^2 \qquad hyperbola$$
$$\beta^2=\alpha^2 \qquad parabola$$
$$\alpha^2=0 \qquad \text{circle}$$
$$\beta^2=0 \qquad horizontal\ line$$

Discussion and Further Research

The classical trajectories of E16 and S7 are found in the form of explicit equations. These case by case trajectories are plotted nicely, which are extremely important. Their great value for physical application is like the value of the prime numbers for the whole integers. The key thing is that this research provides examples that the superintegrable system can lead to the explicit results. The quantum analogy of the two systems are still open questions. If the quantum case of two systems can be studied well, then E16 and S7 are characterized completely. Also, it's worthwhile to explore the relationship between these 2D 2nd order nondegenerate systems and 2D 2nd order degenerate systems.

Acknowledgement

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